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How to Superize Liouville Equation

Dimitry LEITES

Department of Mathematics, University of Stockholm, Roslagsv. 101,
Kräftriket hus 6, S-106 91, Stockholm, Sweden
E-mail: mleites@matematik.su.se

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Abstract

So far, there are described in the literature two ways to superize the Liouville equation: for a scalar field (for $N \leq 4$) and for a vector-valued field (analogs of the Leznov–Saveliev equations) for $N = 1$. Both superizations are performed with the help of Neveu–Schwarz superalgebra. We consider another version of these superLiouville equations based on the Ramond superalgebra, their explicit solutions are given by Ivanov–Krivonos' scheme. Open problems are offered.

1 A missed superization of the Liouville equation

The Liouville equation

$$\frac{\partial^2}{\partial x \partial y} f = \exp(2f) \tag{1.1}$$

so important in applications ranging from SUGRA to soap manufacturing (see, e.g., [15], [6], [14]), [12] is manifestly invariant with respect to the Witt algebra $\mathfrak{witt}$, the Lie algebra of conformal transformations:

$$x \mapsto a(x),\ y \mapsto b(y),\ f \mapsto f + \ln a + \ln b \text{ for } x, y \in \mathbb{C}. \tag{1.2}$$

This invariance provides us with a "general" (in the sense that it belongs to an open orbit) solution

$$\exp(f) = \frac{{a_x}'{b_y}'}{(1-ab)^2}. \tag{1.3}$$

For super version of (1.1) one invariably takes ([15], [6])

$$D_+ D_- f = m \exp(f),\ \text{ where } D_\pm = \frac{\partial}{\partial \theta_\pm} + \theta_\pm \frac{\partial}{\partial x_\pm}. \tag{1.4}$$

Equation (1.4) was first integrated in [9] in components. The solution in terms of superfields was obtained by Ivanov and Krivonos [7]:

$$\exp(-f) = \frac{D_+(\frac{D_+a}{\sqrt{D_+^2(a)}})D_-(\frac{D_-b}{\sqrt{D_-^2(b)}})}{m^2a - b + m\frac{D_-bD_+a}{\sqrt{D_-^2(b)D_+^2(a)}}}. \tag{1.5}$$

A completely integrable version of (1.4) for a vector-valued function $f = (f_1, \dots f_n)$ (together with a way to get an explicit solution) was given in [10] for the systems (equivalent for invertible matrix A):

$$D_+D_-f_i = \exp(\sum_j A_i^j f_j) \text{ and } D_+D_-f_i = \sum_j A_i^j \exp(f_j), 1 \leq i \leq n \tag{1.6}$$

where A is the Cartan matrix of the Lie superalgebra $\mathfrak{g}(A)$ admitting a superprincipal embedding of $\mathfrak{osp}(1|2)$ (all such embeddings are classified in [10]). Simultaneously, [1], see also [13], observed that a wider class of Cartan matrices is worth considering, namely, all algebras $\mathfrak{g}(A)$ of polynomial growth with a system of simple roots all of which are odd.

The aim of this note is to draw attention to "missed opportunities" in superization of (1.1). Namely, in (1.4)–(1.6) and in the solution of (1.6) we may replace the above $D_\pm$ (which are copies of the contact field $K_{\theta_\pm}$ described in [5] and corresponding to the Neveu-Schwarz algebra) with

$$\tilde{D}_\pm = \frac{1}{2}\tilde{K}_{\theta_\pm} = \theta_\pm \frac{\partial}{\partial t_\pm} - \frac{1}{t_\pm}\frac{\partial}{\partial \theta_\pm} \tag{1.7}$$

which correspond to the Ramond superalgebra. Observe that since the algebras of invariance of the equations corresponding to Neveu–Schwarz and Ramond superalgebras are non-isomorphic, these equations are essentially different though their underlying equations in Bose sector are equivalent (accordingly, the even parts of NS and R are isomorphic).

The above applies also to the superizations of the Brockett equation [18] and Saveliev-Vershik's version [17] of the Liouville equation

$$\frac{\partial^2}{\partial x \partial y} f = K \exp(f) \tag{1.8}$$

associated with the *continuum* Lie superalgebras, cf. [17], i.e., when the non-existent Cartan matrix is replaced with a *nonlinear* operator K. Recently B. Shoikhet and A. Vershik [16] found an explicit form of K for $\mathfrak{gl}(\lambda) = L(U(\mathfrak{sl}(2))/(\Omega - \lambda^2 + 1))$, where Ω is the quadratic Casimir operator of $\mathfrak{sl}(2)$ and $L(A)$ is the Lie algebra constructed on the space of the associative algebra A by replacing the dot product with the bracket, cf. also [4] and [22], where more general Lie (super)algebras based on any simple $\mathfrak{g}$, rather than $\mathfrak{sl}(2)$, are considered.

2 Open problems

1) There are 4 series and several exceptional simple stringy ("superconformal") Lie superalgebras that generalize $\mathfrak{witt}$, see [5] (or [2]). All are realized on the supercircle of

dimension $1|N$. For Lie superalgebras with polynomial coefficients corresponding to some stringy superalgebras with $N \leq 4$ Ivanov and Krivonos suggested a scheme for explicit integration of the analogs of the Liouville equation for the *scalar* field and executed it for small values of N. They did not consider the higher N for two reasons: (a) the volume of calculations is too high to undertake without serious motivations, (b) for $N > 4$ spin > 2. In view of Vasiliev's ideas [22] (b) can be considered as an obsolete and non-existing obstacle, and in view of a new technique [3] the volume of calculations might become tolerable.

What does Ivanov–Krivonos' scheme give for greater N and other stringy superalgebras?

2) How to apply Ivanov–Krivonos' scheme for $N > 1$ to the matrix equations from [10] (or the other way round)? This blend should be associated with a generalization of the "superprincipal embedding". Though its meaning for $N > 1$ is unclear; an approach is suggested in [4].

3) The 12 of the simple stringy Lie superalgebras are *distinguished*: only they have nontrivial central extensions. (Observe that since one of the distinguished superalgebras has 3 nontrivial central extensions each, there are exactly 14 direct superizations of the Schrödinger equation, KdV hierarchies and related structures, see [11].)

Ivanov-Krivonos's scheme seem to require simple Lie superalgebras G of vector fields with polynomial coefficients, not Laurent ones G^L. Unlike G^L, algebras G have no central extension, so the importance of the distinguished stringy superalgebras for integration of the analogs of Liouville equation is unclear. Recently, however, F. Toppan demonstrated [21] (unpublished) that the central charge of the Virasoro algebra $\mathfrak{vir}$, the central extension of $\mathfrak{witt}$, non-trivially acts on (1.1). This miraculous result indicates that probably there are other, inner, mechanisms that restrict application of Ivanov-Krivonos's scheme for non-distinguished stringy Lie superalgebras G^L. What are these mechanisms, if any?

4) I considered the above equations over complex numbers. In applications functions of real variable are often preferable. This leads to necessity to classify real forms of stringy Lie superalgebras. Though this is done for almost all simple stringy superalgebras, see [19] (and the results are vital even for the description of real forms of Kac–Moody algebras, cf. [20]), the answer seems to be unknown to physicists being published in a purely mathematical journal. The description of the real forms of the remaining simple stringy superalgebras is in preparation; to describe all real solutions for higher N and single out physically relevant ones is a problem, cf. [7].

5) (Due to a referee.) One may expect, on purely aesthetic grounds, that in super case there **must** exist some kind of superstructure over the **set** (1.5) of solutions of super Liouville equations. I.e., on a deeper level there must be a "superset" of solutions on which an infinite dimensional supergroup (*not* Lie!) of automorphisms acts, so that the corresponding "superset" is realized as an open superorbit of this supergroup.

To extend the constructs of Ivanov-Krivonos scheme to this level as well, one needs to have, evidently, the super version of Kac–Peterson's theory (cf. [8]) of groups associated with integrable algebras.

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